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Dynamic Response of an Axially Loaded Bending–Torsion Coupled Beam

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A method is presented to predict the response of an axially loaded beam coupled in bending and torsion to deterministic and random loads. The beam is uniform and is assumed to carry a constant axial load. It is subjected to time-dependent bending and/or torsional load, which can be either deterministic or random. Both concentrated and distributed loads are considered. The deterministic load is assumed to vary harmonically, whereas the random load is assumed to be Gaussian, having both stationary and ergodic properties. The theory developed is applied to a cantilever wind turbine blade for which there is substantial coupling between the bending and torsional modes of deformation. Numerical results are presented and the effect of axial load on dynamic flexural and torsional displacements is discussed.

Introduction

THE response of beams to deterministic and random loads has been investigated by many authors.[1-7] Also, there are some excellent texts[8-10] that cover the basic as well as advanced aspects of the theory on the subject. However, the available literature on this topic concentrates either on a Bernoulli–Euler beam,[1,3] a Timoshenko beam,[2] or an axially loaded Timoshenko beam,[6] in which only flexural displacement (uncoupled with torsion) of the beam is considered. Thus, the reported beam theory assumes that there is no coupling between the bending displacement and the torsional rotation. Such an assumption is valid only for simple beams with coincident mass center and shear center (e.g., doubly symmetric cross sections like circle, square, etc.). Naturally, the assumption imposes severe restriction on the dynamic response analysis of beams for which the mass center and shear center are not coincident. The examples of such beams and their applications include aircraft wings, helicopter, turbine blades, and the hull of ships.

The free natural vibration characteristics of a bending–torsion coupled beam, and an axially loaded bending–torsion coupled beam have been investigated, respectively, by Banerjee[11,12] and Banerjee and Fisher[13] in recent years using an exact dynamic stiffness method. It appears that there has not been any attempt made to determine the deterministic and/or random response of such a bending–torsion coupled beam. In this article a theory is developed to predict the flexural and torsional response of an axially loaded bending–torsion coupled beam when acted upon by deterministic as well as random loads. (The response quantities considered in this article are the flexural and torsional displacements at various points along the length of the beam.) Both concentrated and distributed loads are considered. A harmonically varying load is considered for the deterministic case, whereas in the case of random loading, the input is considered to be an ideal white noise having stationary and ergodic properties.

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The following steps are used when developing the theory of this article. First, the natural frequencies and mode shapes in free vibration of an axially loaded bending–torsion coupled beam are obtained using the method of Banerjee and Fisher.[13] A normal mode method is then used to compute the frequency response function of the bending–torsion coupled beam. (Note that linear small deflection theory has been used so that the overall response of the beam is represented by the superposition of all individual responses in each mode.) The Duhamel's integral is employed to calculate the response for the deterministic case. The evaluation of the response for the random load is, however, carried out in the frequency domain by relating the power spectral density (PSD) of the output to that of the input via the modulus of the complex frequency response function. Representative results are given for a wind turbine blade with cantilever end conditions. The effect of axial force on the response characteristics is demonstrated.

Theory

The differential equations of motion of an axially loaded and viscously damped bending–torsion coupled beam (see Fig. 1) are taken in the following form[13]:

$$EI\frac{\partial^4 u}{\partial y^4} + P\left(\frac{\partial^2 u}{\partial y^2} - x_\alpha\frac{\partial^2 \psi}{\partial y^2}\right) + c_1\left(\frac{\partial u}{\partial t} - x_\alpha\frac{\partial \psi}{\partial t}\right) + m\left(\frac{\partial^2 u}{\partial t^2} - x_\alpha\frac{\partial^2 \psi}{\partial t^2}\right) = f(y, t) \quad (1)$$

$$GJ\frac{\partial^2 \psi}{\partial y^2} + \frac{P}{m}\left(mx_\alpha\frac{\partial^2 u}{\partial y^2} - I_\alpha\frac{\partial^2 \psi}{\partial y^2}\right) + \left(mx_\alpha\frac{\partial^2 u}{\partial t^2} - I_\alpha\frac{\partial^2 \psi}{\partial t^2}\right) + \left(c_1 x_\alpha\frac{\partial u}{\partial t} - c_2\frac{\partial \psi}{\partial t}\right) = g(y, t) \quad (2)$$

where $u = u(y, t)$ and $\psi = \psi(y, t)$ are the transverse displacement and the torsional rotation of the flexural axis of the beam (the flexural axis is defined here as the locus of shear centers of the beam cross sections), $f(y, t)$ and $g(y, t)$ are the external force and torque acting on and about the flexural axis of the beam, m is the mass per unit length, EI and GJ are the bending and torsional rigidity of the beam, I_α is the mass moment of inertia per unit length, x_α is the distance between elastic axis and mass axis, and P is a compressive axial load. (Note that P can be negative and, hence, tension is included.)

The coefficients c_1 and c_2 in Eqs. (1) and (2) are linear viscous damping terms per unit length in flexure and torsion,

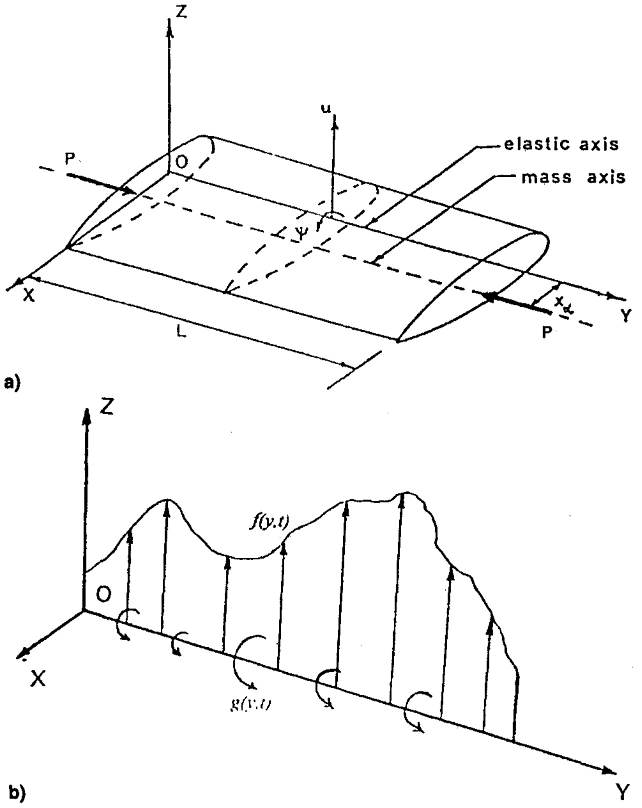

Fig. 1 a) Coordinate system and notation for an axially loaded bending–torsion coupled beam and b) distribution of bending and torsional loads.

respectively. It is assumed that each point of the cross section moves with a different local velocity, so that in Eq. (1), the local damping force sums over the section to the given expression containing the c_1 term. Similarly, in Eq. (2), the expression containing the c_2 term is a torque about the elastic axis because of the elemental damping forces. No other sources of damping are taken into account.

Free Vibration Analysis

For undamped free vibration, the external load $f(y, t)$ and external torque $g(y, t)$ are set to zero together with the damping coefficients. The solutions are then assumed in the form

$$u(y, t) = U_n(y)e^{i\omega_n t} \tag{3}$$

$$\psi(y, t) = \Psi_n(y)e^{i\omega_n t} \tag{4}$$

where $n = 1, 2, 3, \ldots, i = \sqrt{-1}$.

Substituting Eqs. (3) and (4) into Eqs. (1) and (2) with $c_1 = c_2 = f(y, t) = g(y, t) = 0$ gives the following two simultaneous differential equations for U_n and Ψ_n:

$$EI\frac{d^4U_n}{dy^4} + P\left(\frac{d^2U_n}{dy^2} - x_\alpha\frac{d^2\Psi_n}{dy^2}\right) - m\omega_n^2(U_n - x_\alpha\Psi_n) = 0 \tag{5}$$

$$GJ\frac{d^2\Psi_n}{dy^2} + \frac{P}{m}\left(mx_\alpha\frac{d^2U_n}{dy^2} - I_\alpha\frac{d^2\Psi_n}{dy^2}\right) - \omega_n^2(mx_\alpha U_n - I_\alpha\Psi_n) = 0 \tag{6}$$

By eliminating either Ψ_n or U_n from Eqs. (5) and (6), we obtain

$$(D^6 + aD^4 - bD^2 - c)W_n = 0 \tag{7}$$

where

$$W_n = U_n \text{ or } \Psi_n, \quad D = \frac{d}{d\xi}, \quad \xi = \frac{y}{L}$$

$$a = \frac{[\bar{a}\bar{b} + \bar{p}(\bar{b} - \bar{a}\bar{p}\bar{c})]}{(\bar{b} - \bar{a}\bar{p})} \tag{8}$$

$$b = \frac{\bar{b}(\bar{b} - 2\bar{a}\bar{p}\bar{c})}{(\bar{b} - \bar{a}\bar{p})}, \quad c = \frac{\bar{a}\bar{b}^2\bar{c}}{(\bar{b} - \bar{a}\bar{p})}$$

with

$$\bar{p} = PL^2/EI, \quad \bar{a} = I_\alpha\omega_n^2L^2/GJ, \quad \bar{b} = m\omega_n^2L^4/EI, \quad \bar{c} = 1 - mx_\alpha^2/I_\alpha \tag{9}$$

where L is the length of the beam. The solution of the differential Eq. (7) is[13]

$$W_n(\xi) = D_1\cosh\alpha\xi + D_2\sinh\alpha\xi + D_3\cos\beta\xi + D_4\sin\beta\xi + D_5\cos\gamma\xi + D_6\sin\gamma\xi \tag{10}$$

where

$$\alpha = [2(q/3)^{0.5}\cos(\varphi/3) - a/3]^{0.5}$$
$$\beta = \{2(q/3)^{0.5}\cos[(\pi - \varphi)/3] + a/3\}^{0.5} \tag{11}$$
$$\gamma = \{2(q/3)^{0.5}\cos[(\pi + \varphi)/3] + a/3\}^{0.5}$$

in which

$$q = b + (a^2/3)$$
$$\varphi = \cos^{-1}\{(27c - 9ab - 2a^3)/[2(a^2 + 3b)^{1.5}]\} \tag{12}$$

Equation (10) represents the solution for both the bending displacement $U_n(\xi)$ and torsional rotation $\Psi_n(\xi)$. Thus,

$$U_n(\xi) = A_1\cosh\alpha\xi + A_2\sinh\alpha\xi + A_3\cos\beta\xi + A_4\sin\beta\xi + A_5\cos\gamma\xi + A_6\sin\gamma\xi$$
$$\Psi_n(\xi) = B_1\cosh\alpha\xi + B_2\sinh\alpha\xi + B_3\cos\beta\xi + B_4\sin\beta\xi + B_5\cos\gamma\xi + B_6\sin\gamma\xi \tag{13}$$

where A_1–A_6 and B_1–B_6 are two different sets of constants that are not all independent. Substituting Eqs. (13) into Eqs. (5) and (6) shows that

$$B_1 = (k_\alpha/x_\alpha)A_1, \quad B_3 = (k_\beta/x_\alpha)A_3, \quad B_5 = (k_\gamma/x_\alpha)A_5$$
$$B_2 = (k_\alpha/x_\alpha)A_2, \quad B_4 = (k_\beta/x_\alpha)A_4, \quad B_6 = (k_\gamma/x_\alpha)A_6 \tag{14}$$

where

$$k_\alpha = 1 - \alpha^4/(\bar{b} - \alpha^2\bar{p}), \quad k_\beta = 1 - \beta^4/(\bar{b} + \beta^2\bar{p})$$
$$k_\gamma = 1 - \gamma^4/(\bar{b} + \gamma^2\bar{p}) \tag{15}$$

Equations (13), in conjunction with the boundary conditions, yield the eigenvalues (natural frequencies) and eigenfunctions (mode shapes) of a bending–torsional coupled beam. Based on Eqs. (5) and (6) and the boundary conditions, the following orthogonality condition for different mode shapes is obtained as

$$\int_0^1 [(mU_mU_n + I_\alpha\Psi_m\Psi_n) - mx_\alpha(U_n\Psi_m + U_m\Psi_n)]\,d\xi = \mu_n\delta_{mn} \tag{16}$$

where μ_n is the generalized mass in the nth mode and δ_{mn} is the Kronecker delta function. Note that the orthogonality condition is valid for any (classical) end conditions of the beam, provided the ends $\xi = 0$ and/or $\xi = 1$ of the beam concerned are free, simply supported, or clamped.

Response to Deterministic Loads

For forced vibration problem, $u(y, t)$ and $\psi(y, t)$ can be expressed in terms of the eigenfunctions as follows:

$$u(y, t) = u(\xi L, t) = \sum_{n=1}^{\infty} q_n(t)U_n(y) \tag{17}$$

$$\psi(y, t) = \psi(\xi L, t) = \sum_{n=1}^{\infty} q_n(t)\Psi_n(y) \tag{18}$$

where $q_n(t)$ are the generalized time-dependent coordinates for each mode. Substituting Eqs. (17) and (18) into Eqs. (1) and (2) and using Eqs. (5) and (6), we obtain

$$\sum_{n=1}^{\infty} [(m\omega_n^2 U_n - mx_\alpha\omega_n^2\Psi_n)q_n + mU_n\ddot{q}_n + c_1U_n\dot{q}_n - c_1\Psi_n x_\alpha\dot{q}_n - mx_\alpha\Psi_n\ddot{q}_n] = f(\xi, t) \tag{19}$$

$$\sum_{n=1}^{\infty} [(mx_\alpha\omega_n^2 U_n - I_\alpha\omega_n^2\Psi_n)q_n + mx_\alpha U_n\ddot{q}_n - c_2\Psi_n\dot{q}_n + c_1U_n x_\alpha\dot{q}_n - I_\alpha\Psi_n\ddot{q}_n] = g(\xi, t) \tag{20}$$

where an overdot represents differentiation with respect to time.

Multiplying Eqs. (19) and (20) by U_m and $-\Psi_m$, respectively, then summing up these equations and integrating from 0 to L, and using the orthogonality condition of Eq. (16), gives the following equation:

$$\ddot{q}_n(t) + 2\zeta_n\omega_n\dot{q}_n(t) + \omega_n^2 q_n(t) = [F_n(t) + G_n(t)] \tag{21}$$

where $F_n(t)$ and $G_n(t)$ can be expressed as

$$F_n(t) = \frac{1}{\mu_n}\int_0^L U_n(y)f(y, t)\,\mathrm{d}y = \frac{1}{\mu_n}P_n(t) \tag{22a}$$

$$G_n(t) = -\frac{1}{\mu_n}\int_0^L \Psi_n(y)g(y, t)\,\mathrm{d}y = \frac{1}{\mu_n}Q_n(t) \tag{22b}$$

$P_n(t)$ and $Q_n(t)$ are the generalized forces, and ζ_n is the nondimensional damping coefficient in the nth mode given by

$$\zeta_n = c_1/(2m\omega_n) = c_2/(2mr^2\omega_n) \tag{23}$$

where r is the radius of gyration defined as $\sqrt{(I_\alpha/m)}$. Note that the assumption $c_2 = r^2c_1$ is currently under investigation and follows a recently published paper.[7]

Using Duhamel's integral, the general solution for Eq. (21) can be obtained as follows:

$$q_n(t) = e^{-\zeta_n\omega_n t}[A_n\cos(\omega_{nd}t) + B_n\sin(\omega_{nd}t)] + \frac{1}{\omega_{nd}}\int_0^t [F_n(\tau) + G_n(\tau)]e^{-\zeta_n\omega_n(t-\tau)}\sin[\omega_{nd}(t-\tau)]\,\mathrm{d}\tau \tag{24}$$

where $\omega_{nd} = \omega_n\sqrt{1 - \zeta_n^2}$, and A_n and B_n are the coefficients related to the initial conditions. Finally, substituting Eq. (24) into Eqs. (17) and (18), the general solutions for $u(y, t)$ and $\psi(y, t)$ can be obtained as follows:

$$u(y, t) = \sum_{n=1}^{\infty} U_n(y)\left\{e^{-\zeta_n\omega_n t}[A_n\cos(\omega_{nd}t) + B_n\sin(\omega_{nd}t)] + \frac{1}{\omega_{nd}}\int_0^t [F_n(\tau) + G_n(\tau)]e^{-\zeta_n\omega_n(t-\tau)}\sin[\omega_{nd}(t-\tau)]\,\mathrm{d}\tau\right\} \tag{25}$$

$$\psi(y, t) = \sum_{n=1}^{\infty} \Psi_n(y)\left\{e^{-\zeta_n\omega_n t}[A_n\cos(\omega_{nd}t) + B_n\sin(\omega_{nd}t)] + \frac{1}{\omega_{nd}}\int_0^t [F_n(\tau) + G_n(\tau)]e^{-\zeta_n\omega_n(t-\tau)}\sin[\omega_{nd}(t-\tau)]\,\mathrm{d}\tau\right\} \tag{26}$$

If the external force and torque are assumed as $f(y, t) = \delta(y - a_i)F_i\sin\Omega t$ and $g(y, t) = \delta(y - b_i)G_i\sin\Omega t$, which represent a system of concentrated simple harmonic forces and torques acting at points a_i and b_i, respectively, with $i = 1, 2, \ldots, N$, and $\delta(y)$ is the Dirac delta function; then the dynamic response $u(y, t)$ and $\psi(y, t)$ can be obtained as

$$u(\xi, t) = \sum_{n=1}^{\infty} U_n(\xi)e^{-\zeta_n\omega_n t}[A_n\cos(\omega_{nd}t) + B_n\sin(\omega_{nd}t)] + \sum_{n=1}^{\infty} U_n(\xi)\sum_{i=1}^{N}\left\{\frac{[U_n(a_i)F_i + \Psi_n(b_i)G_i]}{\mu_n[(\omega_n^2 - \Omega^2)^2 + (2\zeta_n\omega_n\Omega)^2]} \times [(\omega_n^2 - \Omega^2)\sin(\Omega t) - (2\zeta_n\omega_n\Omega)\cos(\Omega t)]\right\} \tag{27}$$

$$\psi(\xi, t) = \sum_{n=1}^{\infty} \Psi_n(\xi)e^{-\zeta_n\omega_n t}[A_n\cos(\omega_{nd}t) + B_n\sin(\omega_{nd}t)] + \sum_{n=1}^{\infty} \Psi_n(\xi)\sum_{i=1}^{N}\left\{\frac{[U_n(a_i)F_i + \Psi_n(b_i)G_i]}{\mu_n[(\omega_n^2 - \Omega^2)^2 + (2\zeta_n\omega_n\Omega)^2]} \times [(\omega_n^2 - \Omega^2)\sin(\Omega t) - (2\zeta_n\omega_n\Omega)\cos(\Omega t)]\right\} \tag{28}$$

Equations (27) and (28) provide the general solutions for harmonically varying multipoint force and torque loading at given locations. If there is only one external force and only one external torque acting on the beam at $y = a_1$ and $y = b_1$, respectively, then $N = 1$ and $f(y, t) = \delta(y - a_1)F\sin\Omega t$ and $g(y, t) = \delta(y - b_1)G\sin\Omega t$. Assuming the initial conditions to be zero, the steady-state dynamic response can then be written as

$$u(\xi, t) = \sum_{n=1}^{\infty} U_n(\xi)[U_n(a_1)F + \Psi_n(b_1)G] \times \left\{\frac{[(\omega_n^2 - \Omega^2)\sin\Omega t - (2\zeta_n\omega_n\Omega)\cos\Omega t]}{\mu_n[(\omega_n^2 - \Omega^2)^2 + (2\zeta_n\omega_n\Omega)^2]}\right\} \tag{29}$$

$$\psi(\xi, t) = \sum_{n=1}^{\infty} \Psi_n(\xi)[U_n(a_1)F + \Psi_n(b_1)G] \times \left\{\frac{[(\omega_n^2 - \Omega^2)\sin\Omega t - (2\zeta_n\omega_n\Omega)\cos\Omega t]}{\mu_n[(\omega_n^2 - \Omega^2)^2 + (2\zeta_n\omega_n\Omega)^2]}\right\} \tag{30}$$

Equations (29) and (30) can be simplified as

$$u(\xi, t) = \sum_{n=1}^{\infty} U_n(\xi)[U_n(a_1)F + \Psi_n(b_1)G]\frac{A_n}{\mu_n\omega_n^2}\sin(\Omega t - \varphi) \tag{31}$$

$$\psi(\xi, t) = \sum_{n=1}^{\infty} \Psi_n(\xi)[U_n(a_1)F + \Psi_n(b_1)G] \frac{A_n}{\mu_n \omega_n^2} \sin(\Omega t - \varphi) \tag{32}$$

where

$$\tan \varphi = \frac{2\zeta_n \Omega/\omega_n}{1 - \Omega^2/\omega_n^2} \tag{33}$$

$$A_n = [(1 - \Omega^2/\omega_n^2)^2 + (2\zeta_n\Omega/\omega_n)^2]^{-0.5} \tag{34}$$

This completes the theoretical analysis of the response of the axially loaded bending–torsion coupled beam to deterministic loads.

Response to Random Loads

The receptances $H_u(\xi, \xi_1, \Omega)$ and $H_\psi(\xi, \xi_1, \Omega)$ for the bending displacement u, and the torsional rotation ψ, respectively, are defined by their amplitudes at the point ξ when a harmonically varying force and/or torque with amplitude 1 and circular frequency Ω is applied at ξ_1. Thus, for the purpose of computing receptances, the externally applied loading $f(\xi, t)$ and $g(\xi, t)$ are represented by

$$f(\xi, t) = \delta(\xi - \xi_1)e^{i\Omega t}$$
$$g(\xi, t) = \delta(\xi - \xi_1)e^{i\Omega t} \tag{35}$$

Substituting from Eqs. (35) into Eqs. (22) gives

$$F_n(t) = (1/\mu_n)U_n(\xi_1)e^{i\Omega t}$$
$$G_n(t) = -(1/\mu_n)\Psi_n(\xi_1)e^{i\Omega t} \tag{36}$$

The solution of Eq. (21) for the previous loading is taken in the form

$$q_n(t) = q_{on}e^{i\Omega t} \tag{37}$$

Substituting Eq. (37) into Eq. (21) and using Eqs. (36), we obtain

$$q_{on} = \frac{V_n(\xi_1)}{\mu_n(\omega_n^2 - \Omega^2 + 2i\zeta\Omega\omega_n)} \tag{38}$$

in which $V_n(\xi_1) = a_F U_n(\xi_1) - a_G \Psi_n(\xi_1)$, where a_F (or a_G) is equal to 1 or 0, depending upon whether applied transverse force (or torque) is presented or not. The receptance for u and ψ can now be obtained from Eqs. (17) and (18) with the help of Eq. (38) as follows:

$$H_u(\xi, \xi_1, \Omega) = \sum_{n}^{\infty} q_{on}(\xi_1, \Omega)U_n(\xi)$$
$$H_\psi(\xi, \xi_1, \Omega) = \sum_{n}^{\infty} q_{on}(\xi_1, \Omega)\Psi_n(\xi) \tag{39}$$

Once the receptance or the complex frequency response function is known, the response to a stationary, ergodic random load can be found by following the standard procedure.[6]

The cross-spectral densities $S_f(\xi_1, \xi_2, \Omega)$ and $S_g(\xi_1, \xi_2, \Omega)$, and the cross-correlation functions $R_f(\xi_1, \xi_2, \Omega)$ and $R_g(\xi_1, \xi_2, \Omega)$ of the input excitation are related by their Fourier transform pair as[8–10]

$$S_f(\xi_1, \xi_2, \Omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_f(\xi_1, \xi_2, \tau)e^{-i\Omega\tau}\, d\tau$$
$$R_f(\xi_1, \xi_2, \tau) = \int_{-\infty}^{\infty} S_f(\xi_1, \xi_2, \Omega)e^{i\Omega\tau}\, d\Omega$$
$$S_g(\xi_1, \xi_2, \Omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_g(\xi_1, \xi_2, \tau)e^{-i\Omega\tau}\, d\tau$$
$$R_g(\xi_1, \xi_2, \tau) = \int_{-\infty}^{\infty} S_g(\xi_1, \xi_2, \Omega)e^{i\Omega\tau}\, d\Omega \tag{40}$$

The cross-correlation functions $R_f(\xi_1, \xi_2, \tau)$ and $R_g(\xi_1, \xi_2, \tau)$ of the excitation are defined as[8–10]

$$R_f(\xi_1, \xi_2, \tau) = \langle f(\xi_1, t)f(\xi_2, t + \tau)\rangle \tag{41a}$$
$$R_g(\xi_1, \xi_2, \tau) = \langle g(\xi_1, t)g(\xi_2, t + \tau)\rangle \tag{41b}$$

where $\langle\,\rangle$ denotes the ensemble average of the stochastic process.

When the beam is acted upon by a finite number K of concentrated, randomly varying forces and torques, the spectral density function of the bending displacement and torsional rotation are related to the cross-spectral densities of the forces $S_f^{rs}(\Omega)$ and torques $S_g^{rs}(\Omega)$ as[8–10]

$$S_u(\xi, \Omega) = \sum_{r=1}^{K}\sum_{s=1}^{K} H_u^*(\xi, \xi_r, \Omega)H_u(\xi, \xi_s, \Omega)[S_f^{rs}(\Omega) + S_g^{rs}(\Omega)]$$
$$S_\psi(\xi, \Omega) = \sum_{r=1}^{K}\sum_{s=1}^{K} H_\psi^*(\xi, \xi_r, \Omega)H_\psi(\xi, \xi_s, \Omega)[S_f^{rs}(\Omega) + S_g^{rs}(\Omega)] \tag{42}$$

where * denotes the complex conjugate.

For a distributed load, $S_f^{rs}(\Omega)$ and $S_g^{rs}(\Omega)$ are replaced by $S_f(\xi_1, \xi_2, \Omega)\, d\xi_1\, d\xi_2$ and $S_g(\xi_1, \xi_2, \Omega)\, d\xi_1\, d\xi_2$, and the summations are replaced by integrals over the beam length. Thus, for a distributed load, the response spectral densities are given by

$$S_u(\xi, \Omega) = \int_0^1\int_0^1 H_u^*(\xi, \xi_1, \Omega)H_u(\xi, \xi_2, \Omega)[S_f(\xi_1, \xi_2, \Omega) + S_g(\xi_1, \xi_2, \Omega)]\, d\xi_1\, d\xi_2$$
$$S_\psi(\xi, \Omega) = \int_0^1\int_0^1 H_\psi^*(\xi, \xi_1, \Omega)H_\psi(\xi, \xi_2, \Omega)[S_f(\xi_1, \xi_2, \Omega) + S_g(\xi_1, \xi_2, \Omega)]\, d\xi_1\, d\xi_2 \tag{43}$$

Substituting the receptance H_u and H_ψ from Eqs. (39) into Eqs. (43) gives

$$S_u(\xi, \Omega) = \sum_{m}^{\infty}\sum_{n}^{\infty} d_m^*(\Omega)d_n(\Omega)\eta_{mn}(\Omega)U_m(\xi)U_n(\xi)$$
$$S_\psi(\xi, \Omega) = \sum_{m}^{\infty}\sum_{n}^{\infty} d_m^*(\Omega)d_n(\Omega)\eta_{mn}(\Omega)\Psi_m(\xi)\Psi_n(\xi) \tag{44}$$

where

$$d_n(\Omega) = 1/[\mu_n(\omega_n^2 - \Omega^2 + 2i\zeta\Omega\omega_n)] \tag{45}$$

$$\eta_{mn}(\Omega) = \int_0^1 \int_0^1 V_m(\xi_1)V_n(\xi_2)[S_f(\xi_1, \xi_2, \Omega) + S_g(\xi_1, \xi_2, \Omega)]\, d\xi_1\, d\xi_2 \tag{46}$$

The mean square value of the response can now be found by integrating the spectral density functions over all frequencies, so that

$$\langle u^2(\xi, t)\rangle = \int_{-\infty}^{\infty} S_u(\xi, \Omega)\, d\Omega \tag{47}$$

$$\langle \psi^2(\xi, t)\rangle = \int_{-\infty}^{\infty} S_\psi(\xi, \Omega)\, d\Omega \tag{48}$$

If the input random excitation follows a Gaussian probability distribution, the response probability will also be Gaussian and therefore, the response can be fully described by its spectral density function.

Two specific loading cases are considered: a randomly varying force acting at $\xi = \xi_f$ and a randomly varying distributed force.

For the concentrated random load at $\xi = \xi_f$, Eq. (46) gives

$$\eta_{mn}(\Omega) = U_m(\xi_f)U_n(\xi_f)S_f(\Omega) \tag{49}$$

The spectral density of the response is given by Eq. (44) as

$$S_u(\xi, \Omega) = \sum_m^{\infty} \sum_n^{\infty} d_m^*(\Omega)U_m(\xi)U_m(\xi_f)d_n(\Omega)U_n(\xi)U_n(\xi_f)S_f(\Omega)$$

$$= \left| \sum_n^{\infty} d_n(\Omega)U_n(\xi)U_n(\xi_f) \right|^2 S_f(\Omega) \tag{50}$$

If the damping is small and the eigenfrequencies are well separated, Eq. (50) can be approximated by[6]

$$S_u(\xi, \Omega) = \sum_n^{\infty} |d_n(\Omega)U_n(\xi)U_n(\xi_f)|^2 S_f(\Omega) \tag{51}$$

For the case of distributed loading, the randomly varying exciting force can be written as

$$f(\xi, t) = f(\xi)\theta(t) \tag{52}$$

where $\theta(t)$ is a stochastic process. The cross-correlation of the loading $f(\xi, t)$ is then given by

$$R_f(\xi_1, \xi_2, \tau) = \langle f(\xi_1)\theta(t)f(\xi_2)\theta(t+\tau)\rangle = f(\xi_1)f(\xi_2)R_\theta(\tau) \tag{53}$$

The corresponding cross-spectral density is

$$S_f(\xi_1, \xi_2, \Omega) = f(\xi_1)f(\xi_2)S_\theta(\Omega) \tag{54}$$

Equation (46) in this case gives

$$\eta_{mn}(\Omega) = \int_0^1 f(\xi_1)U_m(\xi_1)\, d\xi_1 \int_0^1 f(\xi_2)U_n(\xi_2)\, d\xi_2 S_\theta(\Omega)$$

$$= f_m f_n S_\theta(\Omega) \tag{55}$$

Equations (44) and (55) together give

$$S_u(\xi, \Omega) = \left| \sum_n^{\infty} f_n(\xi)d_n(\Omega)U_n(\xi) \right|^2 S_\theta(\Omega) \tag{56}$$

Similar expressions can be found for S_ψ to give the cross-spectral density of the torsional displacement of the cross section.

If the input random load is assumed to be an ideal white noise, $S_f(\Omega)$ and $S_\theta(\Omega)$ in Eqs. (51) and (56), respectively, can be replaced by a constant, so that

$$S_f(\Omega) = S_\theta(\Omega) = S_0 = \text{const} \tag{57}$$

Results and Discussion

Based on the theory developed in this article, results are obtained for an axially loaded bending–torsion coupled beam with cantilever end conditions. The following data of a wind-turbine blade[14] are used (cross sectional and other properties correspond to the root): 1) $EI = 2.2101 \times 10^7$ nm², 2) $GJ = 5.1483 \times 10^6$ nm², 3) $m = 112.0$ kg/m, 4) $I_\alpha = 21.8$ kg m, 5) $x_\alpha = 0.153$ m, and 6) $L = 12$ m. The damping coefficient ζ_n [see Eq. (23)] is taken to be 0.01 for all modes.

Three load levels are considered. These are $P = 190{,}000$ N (which is 50% of the lowest critical buckling load of the can-

Table 1 Natural frequencies of an axially loaded cantilever turbine blade with different axial loads

Frequency no.	Frequency, rad/s		
	$P = 190$ kN	$P = 0$	$P = -380$ kN
1	7.8	10.8	14.8
2	62.4	64.4	66.8
3	70.1	71.2	74.6
4	177.3	179.5	183.7
5	211.4	212.3	214.1

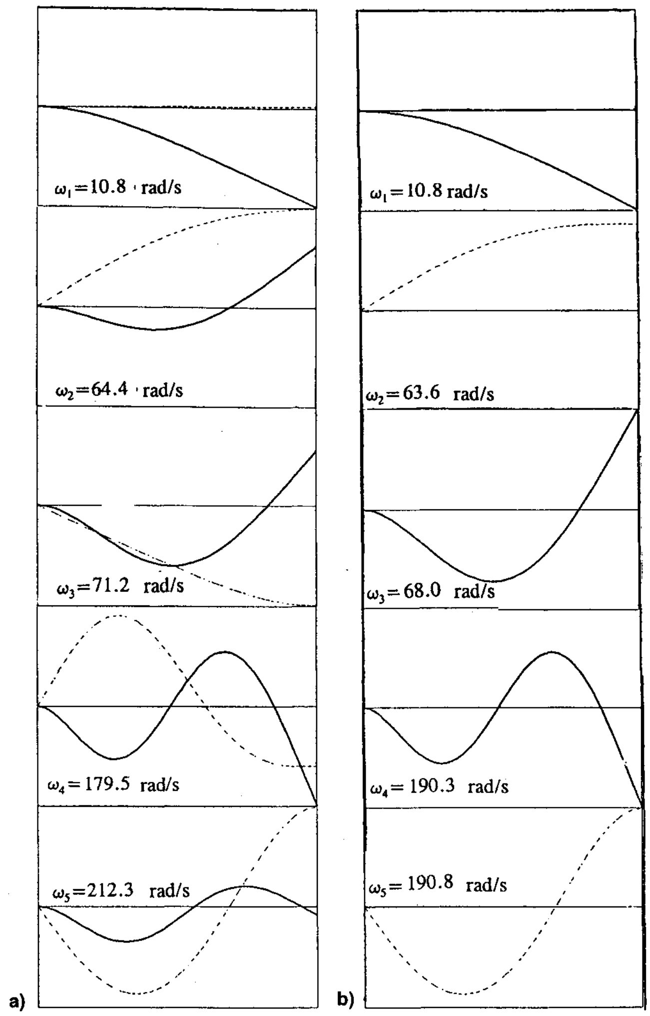

Fig. 2 Natural frequencies and mode shapes of a turbine blade with $P = 0$; —— flexural displacement u, ---- torsional rotation ψ: a) bending–torsion coupled beam theory ($x_\alpha \neq 0$) and b) Bernoulli–Euler beam theory ($x_\alpha = 0$).

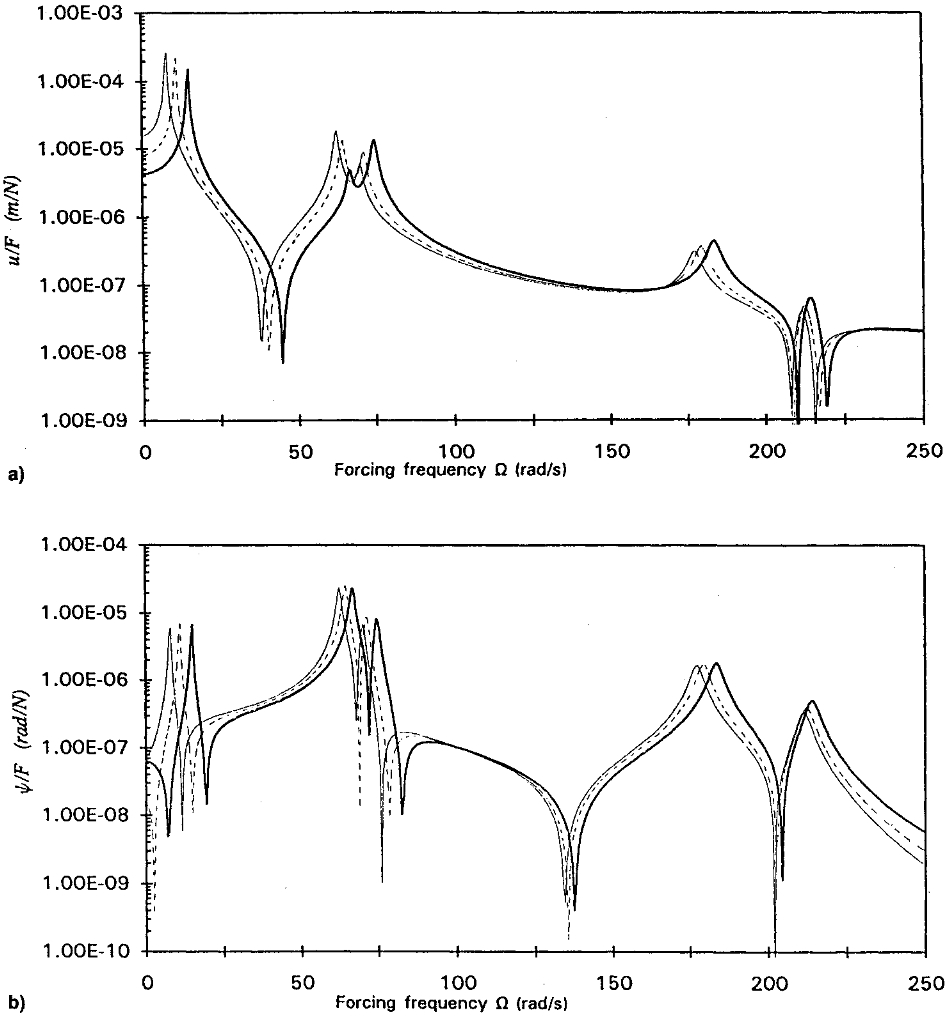

Fig. 3 Dynamic a) flexural and b) torsional displacement at midspan because of a unit harmonically varying concentrated force at the tip. —— compressive, ---- unloaded, —— tensile.

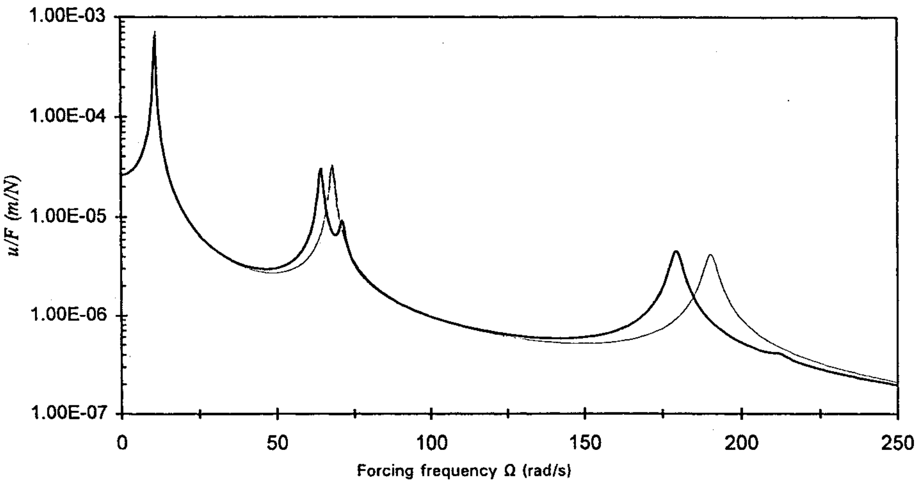

Fig. 4 Dynamic flexural displacement at the tip because of a unit harmonically varying concentrated force at the same point. —— present theory, —— Bernoulli–Euler theory.

tilever), $P = 0$ and $P = -380{,}000$ N (tension). The first five natural frequencies of the blade are shown in Table 1. Natural frequencies are quite well separated and the effect of axial load is also noticeable. The presence of the compressive load has reduced the natural frequencies as expected; for instance, the difference in results when compared with the unloaded case ($P = 0$) is 28% in the first natural frequency and 3% in the second natural frequency. In contrast, the presence of the tensile load has predictably increased the natural frequencies by 37% in the first mode and 3.7% in the second mode. For all loading cases, substantial coupling exists between bending displacements and torsional rotations in the free vibrational modes of the blade, except for the fundamental one. This is evident from representative results for $P = 0$, shown in Fig.

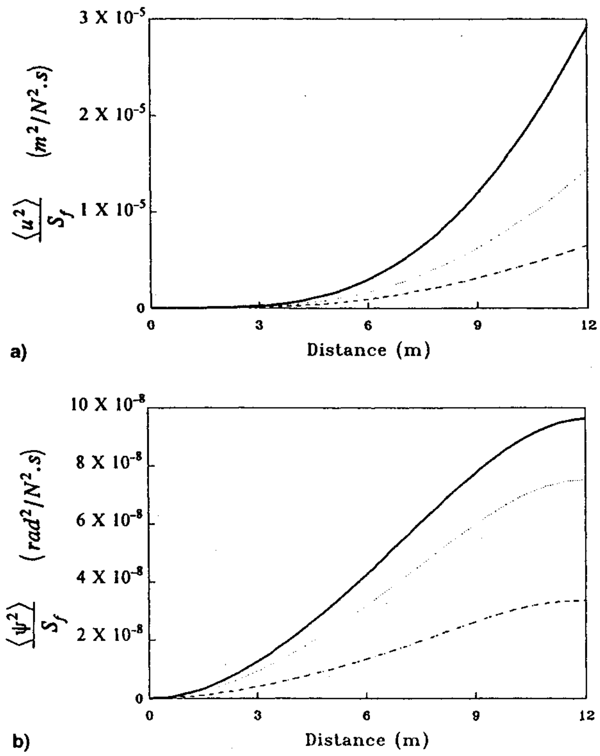

Fig. 5 Variation of mean square value of a) flexural and b) torsional displacement along the blade for different levels of axial load. ——— compressive, ---- unloaded, ······· tensile.

2a. The corresponding natural frequencies and mode shapes for the degenerate case of the Bernoulli–Euler beam, in which the bending–torsion coupling effect is ignored (i.e., when $x_\alpha = 0$ is substituted in the present theory) are shown in Fig. 2b, which clearly indicates torsion-free bending and bending-free torsion as expected. The difference between the two sets of results in Figs. 2a and 2b is quite noticeable, particularly for higher modes.

The dynamic flexural and torsional displacement at the midspan of the blade because of a harmonically varying concentrated flexural force of unit amplitude applied at the tip, for three different levels of axial load, are computed using the present theory and are shown in Figs. 3a and 3b, respectively. The peaks in Figs. 3a and 3b correspond to the natural frequencies of the blade, as expected. Note that the results shown in these figures are obtained under the action of flexural load only, but a dynamic torsional rotation is evident as a consequence of the bending–torsion coupling effect.

To compare results obtained from the present theory with those given by the Bernoulli–Euler theory, the dynamic flexural displacement at the tip of the blade because of a unit harmonically varying concentrated force acting at the same point, was calculated using the modes shown in Figs. 2a and 2b, respectively. The difference between the two sets of results obtained from the two theories is illustrated in Fig. 4. The response of the blade in the fundamental mode is slightly altered, whereas the response behavior in higher modes predicted by the present theory is quite different from that predicted by the Bernoulli–Euler theory. This difference can be attributed to the fact that the fundamental mode of the blade is predominantly a bending mode with torsional coupling (deformation) being almost nonexistent (see Figs. 2a and 2b), whereas for all other modes there is significant coupling between bending displacement and torsional rotation.

In the case of random loading, the externally applied force is assumed to be uniformly distributed as an ideal white noise over the blade length. Also, it is assumed to act only in the flexural direction. The mean square values of the flexural displacement and torsional rotation from root to tip of the blade, for three different levels of axial load obtained from the present theory, are shown in Figs. 5a and 5b. As in the case of deterministic load, the torsional response is induced by modal coupling only. The use of the Bernoulli–Euler or Timoshenko beam theory would not predict such a response. For both sets of loadings, it was found that the first five normal modes were sufficient to describe the response of the particular problem investigated.

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